A Hierarchical Model for the Reliability of an Anti-aircraft Missile System

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Abstract

We describe a hierarchical model for assessing the reliability of multi-component systems. Novel features of this model are the natural manner in which failure time data collected at either the component or subcomponent level is aggregated into the posterior distribution, and pooling of failure information between similar components. Prior information is allowed to enter the model in the form of actual point estimates of reliability at nodes, or in the form of prior groupings. Censored data at all levels of the system are incorporated in a natural way through the likelihood specification. The methodology is illustrated with an example from an anti-aircraft missile system.

1 Background

Reliability of complex systems such as missile systems and automotive systems is a challenging statistical problem. Perhaps the most difficult aspect of *system* reliability assessments is the integration of component, subsystem and system data and prior expert opinion. Usually of interest is determination of how the reliability changes over time. This information is often used to make statistical prediction of system reliability to determine warranty (in the case of automotive systems) and/or shelf-life (in the case of missile systems). While much attention has been paid to *theoretical* system reliability and *empirical* component reliability, there are few instances where the two have been combined for full system *empirical* system reliability when data have been collected on components (or subcomponents) and the full system. In this paper we address this issue central to most major reliability problems by addressing two important analytical concerns: (1) the integration of available information at various levels to assess system reliability and (2) estimating reliability growth or degradation. Methodology for integrating available information in a consistent fashion has proven problematic, and this paper describes a Bayesian hierarchical model that resolves this difficulty. Johnson *et al* (2003) present a general approach to this problem for components which are judged to be successes or failures. In this paper we generalize the approach presented there to allow for continuous failure times.

We present a self-consistent (one that addresses data at different levels and reconciles that with the system structure) model for system reliability. In Section 2, we propose a model for system reliability estimation allowing component, subsystem, and full system testing. That model is illustrated with an application to anti-aircraft missile system data in Section 3. We conclude with a summary of results and suggestions for future work in Section 4.

2 Model

To illustrate the baseline model, consider a system composed of three subsystems. The methods presented here allow for additional levels of granularity in system structure. In general, we assume that failure time data and prior expert opinion are available at different levels, and that our primary goal in modeling such systems is the evaluation of the reliability function, $R_S(t|\Theta_S)$ (where Θ is possibly vector valued), the probability that the system will function beyond time t, $R_S(t|\Theta_S) = Pr(T_S \geq t|\Theta_S)$. Because the system is made up of several components or subsystems, we employ the use of a subscript S to indicate system reliability versus the reliability of a component ($R_2(t|\Theta_2)$ indicates the reliability function of component C_2 or the missile round, for example). Note that this may be specified by the component/subsystem failure time distributions and the system structure as will be demonstrated in our anti-aircraft missile example in Section 3.

This is of interest because it indicates whether the reliability exhibits increasing or decreasing failure rates (indicative of lifecycle position). We will demonstrate that the Bayesian approach presented here estimates a distribution for the hazard function, allowing estimation of the probability of an increasing or decreasing failure rate.

Several sources of information relevant to estimating system reliability are considered. The first is failure time data collected from actual component or subsystem tests. The second source of information takes the form of expert opinion regarding the reliability at time t. A third, less precise source of information is expert opinion regarding the similarity of reliabilities of groups of components within the system or across different systems. For example, in the missile system depicted above, an expert may assert that the reliability of the battery coolant unit (BCU) is similar to the reliability of a BCU in a related missile system, or that reliabilities of the missile round and BCU are similar. However, the expert may not have knowledge regarding the specific probability that any component within a group of similar components functions. Finally, we incorporate the statistical notion that terminal nodes (i.e., components in the reliability block diagram having no subcomponents themselves) may also be grouped into sets of comparably reliable components without the guidance of actual expert opinion. In the baseline model, such information is modeled via an exchangeability assumption on the parameters of the failure time distribution.

To model these sources of information, we first assume that the failure time distributions of components in distinct branches of the reliability block diagram are conditionally independent, and that the success of the system requires successful functioning of all components. Extensions to systems that include redundant components, or in which component failures are not independent, are discussed in the summary. Nodes in the reliability diagram are labeled C_i, where i indicates the component or subcomponent index. The function $a(i)$ provides the parent component (or system) containing (sub)component i, while $g(i, m)$ indicates the group of components that expert m asserts have similar failure rates to component i. We let $R_i(t|\Theta_i)$ denote the reliability of component C_i at time t. The set of components for which test data is available is denoted by S_0, and within this set x_{ij} is the failure time of the j^{th} test of component i, where $j = 1, \ldots, n_i$.

Finally, for terminal nodes in the reliability block diagram, a hierarchical prior specification may be obtained by further assuming that each terminal node's failure time distribution parameters are drawn from a common underlying distribution. For notational simplicity, we assume that the parameters of the failure time distribuions of the terminal nodes are, *a priori*, exchangeable, but this restriction may be relaxed by using expert judgment to group the terminal nodes. If components have different failure time distributions (for example, one component has a Weibull distribution and another component has a lognormal distribution) then this hierarchical specification is inappropriate and strength on terminal nodes will only be borrowed if the components have the same failure time distribution.

As discussed in the previous section, combining data and prior information at different levels within a reliability diagram has often proven problematic, both from the perspectives of computational tractability and model consistency. Our solution to this conundrum is to simply re-express non-terminal node probabilities in terms of terminal node probabilities using deterministic relations derived from an examination of the system reliability diagram. For example, the system diagram suggests that the cumulative distribution function of the system is

$$\begin{aligned} F_S(t|\Theta_S) &= 1 - \prod_{i=2}^{4}[1 - F_i(t|\Theta_i)] \\ &= 1 - \prod_{i=2}^{4} R_i(t|\Theta_i), \end{aligned}$$

where $R_i(t|\Theta_i)$ is the reliability function for the i^{th} component. Applying the chain rule for differentiation, we find that the sampling distribution of the system failure times is induced by

$$f_S(t|\Theta_S) = \sum_{i=2}^{4} f_i(t|\Theta_i) \prod_{j \neq i} (1 - F_j(t|\Theta_i)). \tag{1}$$

This is illustrated with failure time data from an anti-aircraft missile system in Section 3.

Combining these assumptions leads to a joint posterior distribution on the baseline model parameters proportional to

$$\begin{aligned}
[\Theta_S, \Theta_1, \ldots, \Theta_4 | \mathbf{T_S}, \mathbf{T_1}, \ldots \mathbf{T_4}, \eta, \zeta] \quad \propto \quad & \prod_{i=1}^{n_S} \sum_{j=2}^{4} f_j(t_i|\Theta_j) \prod_{k \neq j} R_k(t_i|\Theta_k) \times \\
\times \quad & \prod_{i=1}^{n_2} f_2(t_i|\Theta_2) \times \prod_{i=1}^{n_3} f_3(t_i|\Theta_3) \\
\times \quad & \prod_{i=1}^{n_4} f_4(t_i|\Theta_4) \times \prod_{i=2}^{4} \pi(\Theta_i|\eta) \\
\times \quad & \prod \pi(\eta)
\end{aligned} \tag{2}$$

where $\pi(\Theta_i|\eta)$ is the hierarchical prior specificiation of the parameters for the terminal node failure time distributions. In (2), values of non-terminal node probabilities are assumed to be expressed in terms of the appropriate functions of terminal node probabilities, as defined from the system fault diagram.

3 Example

As a simple demonstration of the proposed methodology, consider a system consisting of only three components which are all required to work in order for the system as a whole to work. Therefore, there are actually four reliability functions of interest, one for each of the three components and one additional reliability function which is the system reliability function. Furthermore, suppose that at each component we conduct $n_i = 20, i = 2, \ldots, 4$ tests and record the time until failure. We also collect $n_S = 10$ full system tests independent of the component data and observe the time until failure. Given this system structure and the test data, we can explore the features of the proposed Bayesian system reliability modeling.

Failure times from the anti-aircraft missile system were observed at both the system level and subsystem level. We observed 10 system tests, and 20 from each of the subsystems. Goodness-of-fit techniques revealed that a reasonable model for the distribution of failure times of the subsystems is Weibull, that is

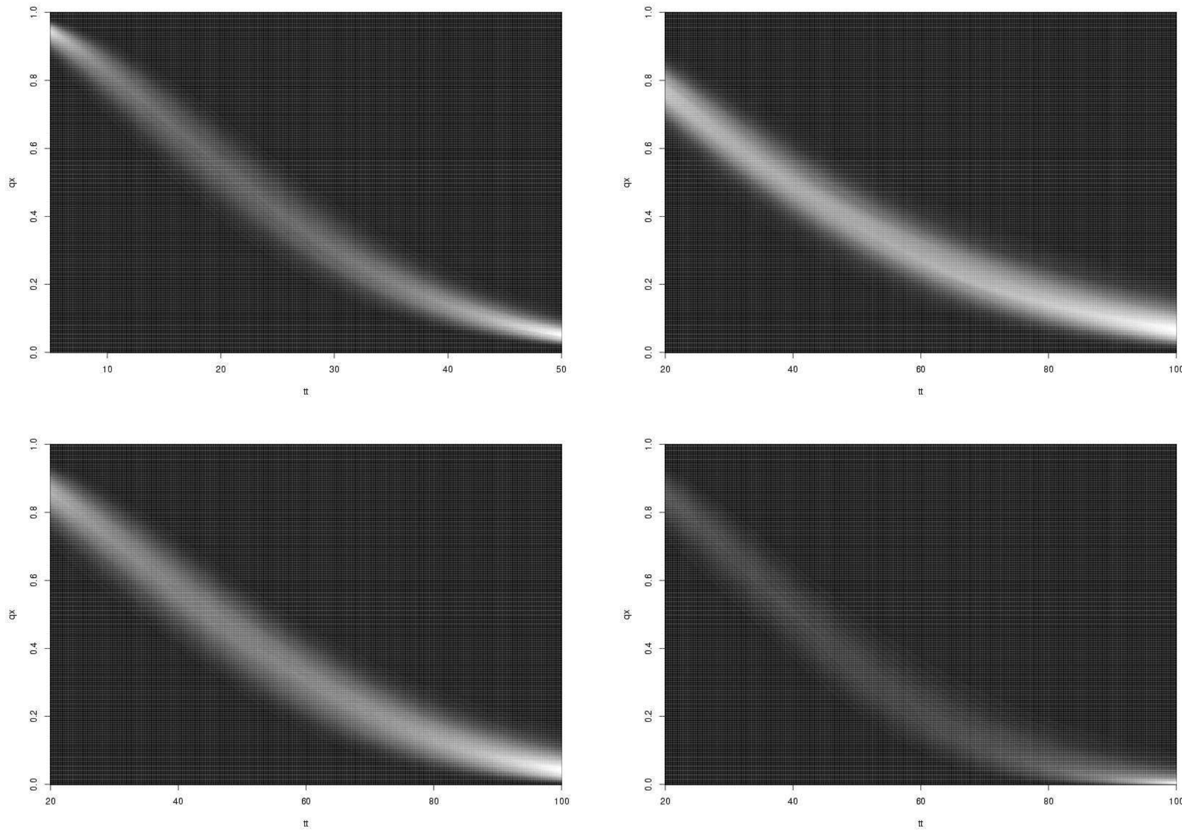
$$f_i(t) = \frac{\alpha_i}{\beta_i} \left(t/\beta_i\right)_i^{\alpha} \exp\left[-\left(t/\beta_i\right)_i^{\alpha}\right], \qquad i = 2, 3, 4$$

so that $\Theta_i = (\alpha_i, \beta_i)$. Our prior specification for Θ_i $(\pi(\Theta|\eta))$ in the example is that the α_i and β_i are all exchangeable and are from a common gamma distribution, that is,

$$\begin{aligned}
\pi(\alpha_i|\lambda_a, \zeta_a) \quad &\propto \quad \alpha_i^{\lambda_a - 1} \exp\left(-\zeta_a \alpha_i\right) \\
\pi(\beta_i|\lambda_b, \zeta_b) \quad &\propto \quad \alpha_i^{\lambda_b - 1} \exp\left(-\zeta_b \alpha_i\right).
\end{aligned}$$

Then, to complete the hierarchical specification, we propose that $\lambda_a, \zeta_a, \lambda_b, \zeta_b$ have exponential distributions, each with its own parameter.

Given the specification above, we use a successive substitution Markov chain Monte Carlo (MCMC) procedure (Gelfand and Smith 1990) where each component of the joint posterior distribution was updated one-at-a-time. The posterior distributions that are presented below were based on 1,000,000 draws from the joint posterior distribution with a 100,000 burn-in period. We did not employ thinning as the convergence diagnostics of Raftery and Lewis (1995) revealed that autocorrelation was not a major problem.



(c) *(d)*

Figure 1: Posterior distributions (as a function of time) for the reliability function of each of the components in the anti-aircraft missile system. They are organized as *(a)*, the posterior distribution of the full system C_1, *(b)* is the posterior distribution for the missile round reliability, *(c)* is the posterior distribution for the BCU, and *(d)* is the posterior distribution for the unnamed component C_5.

The posterior distributions (as a function of time) for the reliability function of each of the components in the example system are presented in Figure 1. They are organized as *(a)*, the posterior distribution of the full system C_1, *(b)* is the posterior distribution for the missile round reliability, *(c)* is the posterior distribution for the BCU, and *(d)* is the posterior distribution for the unnamed component C_5. These plots

4 Conclusions

The proposed hierarchical model offers several advantages over existing models for system reliabilities. Among these are the ease of including diverse sources of information at different levels of the system in the model for overall system reliabilities, a coherent framework for incorporating multiple sources of prior expert opinion through the treatment of expert opinion as (imprecisely-observed) data, and the natural elimination of aggregation errors through the definition of non-terminal probabilities using the assumed structure of the system reliability block diagram and terminal node failure time distributions.

References

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